

THE SZLENK INDEX OF $L_p(X)$ AND A_p

RYAN M. CAUSEY

ABSTRACT. Given a Banach space X , a w^* -compact subset of X^* , and $1 < p < \infty$, we provide an optimal relationship between the Szlenk index of K and the Szlenk index of an associated subset of $L_p(X)^*$. As an application, given a Banach space X , we prove an optimal estimate of the Szlenk index of $L_p(X)$ in terms of the Szlenk index of X . This extends a result of Hájek and Schlumprecht to uncountable ordinals. More generally, given an operator $A : X \rightarrow Y$, we provide an estimate of the Szlenk index of the “pointwise A ” operator $A_p : L_p(X) \rightarrow L_p(Y)$ in terms of the Szlenk index of A .

1. INTRODUCTION

Throughout this work, X will be a fixed Banach space and $K \subset X^*$ will be a w^* -compact, non-empty subset. For $1 < p < \infty$, we let K_p denote the w^* -closure in $L_p(X)^*$ of all functions of the form $gh \in L_q(X^*) \subset L_p(X)^*$, where $g : [0, 1] \rightarrow K$ is simple and Lebesgue measurable, and $h \in B_{L_q}$. Recall that these functions act on $L_p(X)$ by $\langle gh, f \rangle = \int_0^1 \langle g(\varpi), f(\varpi) \rangle h(\varpi) d\varpi$ for $f \in L_p(X)$. Note that if $R \geq 0$ is such that $K \subset RB_{X^*}$, $K_p \subset RB_{L_p(X)^*}$, so that K_p is also w^* -compact. If $K = B_{X^*}$, $K_p = B_{L_p(X)^*}$ by the Hahn-Banach theorem. If $A : X \rightarrow Y$ is an operator, then there exists a “pointwise A ” operator $A_p : L_p(X) \rightarrow L_p(Y)$ given by $(A_p f)(\varpi) = A(f(\varpi))$ for all $\varpi \in [0, 1]$. Then if $K = A^*B_{Y^*}$, $K_p = (A_p)^*B_{L_p(Y)^*}$, which follows from the Hahn-Banach theorem. Thus it is natural to examine what relationship exists between K and K_p . In particular, one may ask what relationship exists between the Szlenk indices of these sets. To that end, we obtain the optimal relationship. In what follows, ω denotes the first infinite ordinal.

Theorem 1. *Fix $1 < p < \infty$. Suppose that ξ is an ordinal such that $Sz(K) \leq \omega^\xi$. Then $Sz(K_p) \leq \omega^{1+\xi}$. If K is convex, $Sz(K_p) \leq \omega Sz(K)$. If K is convex and $Sz(K) \geq \omega^\omega$, $Sz(K) = Sz(K_p)$.*

Using the facts stated in the introduction that $K_p = (A_p)^*B_{L_p(Y)^*}$ if $K = A^*B_{Y^*}$, we immediately deduce the following from Theorem 1.

Corollary 2. *Fix $1 < p < \infty$. If $A : X \rightarrow Y$ is an operator and $K = A^*B_{Y^*}$, then $Sz(A_p) \leq \omega Sz(A)$, and if $Sz(A) \geq \omega^\omega$, $Sz(A_p) = Sz(A)$. In particular, A_p is Asplund if and only if A is.*

Applying Corollary 2 to the identity of a Banach space, we extend the result of Hájek and Schlumprecht from [8] to uncountable ordinals.

We recall that K is said to be w^* -fragmentable if for any non-empty subset L of K and any $\varepsilon > 0$, there exists a w^* -open subset U of X^* such that $L \cap U \neq \emptyset$ and $\text{diam}(L \cap U) < \varepsilon$. We recall that K is w^* -dentable if for any non-empty subset L of K and any $\varepsilon > 0$, there exists a w^* -open slice S of X^* such that $L \cap S \neq \emptyset$ and $\text{diam}(L \cap S) < \varepsilon$. We recall that a w^* -open slice is a subset of X^* of the form $\{x^* \in X^* : \text{Re } x^*(x) > a\}$ for some $x \in X$ and $a \in \mathbb{R}$. As mentioned in [5], a consequence of Corollary 2 is that if $Sz(K) \leq \omega^\xi$, then $Sz(K) \leq Dz(K) \leq \omega^{1+\xi}$, where $Dz(K)$ denotes the w^* -dentability index of K . Thus Corollary 2 implies that K is w^* -dentable if and only if it is w^* -fragmentable.

In addition to considering the Szlenk index of a set, one may consider the ξ -Szlenk power type $\mathbf{p}_\xi(L)$ of the set L , which is important in ξ -asymptotically uniformly smooth renormings of Banach spaces and operators. The concept of a ξ -asymptotically uniformly smooth operator was introduced in [6], and further sharp renorming results regarding the ξ -Szlenk power type of an operator were established in [4]. To that end, we have the following.

Theorem 3. *For any ordinal ξ and any $1 < p < \infty$, if $1/p + 1/q = 1$, $\mathbf{p}_{1+\xi}(K_p) \leq \max\{q, \mathbf{p}_\xi(K)\}$.*

In the case that $\xi \geq \omega$ and $\mathbf{p}_\xi(K) \leq p$, $\mathbf{p}_\xi(K) = \mathbf{p}_\xi(K_p)$, in showing that Theorem 3 is sharp in some cases.

The author wishes to thank P.A.H. Brooker, P. Hájek, N. Holt, and Th. Schlumprecht for helpful remarks during the preparation of this work.

2. $L_p(X)$, TREES, SZLENK INDEX, GAMES

2.1. Trees, $\Gamma_{\xi,n}$, $\mathbb{P}_{\xi,n}$, and stablization results. Given a set Λ , we let $\Lambda^{<\mathbb{N}}$ denote the finite, non-empty sequences in Λ . Given two members s, t of $\Lambda^{<\mathbb{N}}$, we let $s \frown t$ denote the concatenation of s and t , $|s|$ denotes the length of s , $s \preceq t$ means s is an initial segment of t , and $s|_i$ denotes the initial segment of s having length i . Given $t \in \Lambda^{<\mathbb{N}}$, we let $[\preceq t] = \{s \in \Lambda^{<\mathbb{N}} : s \preceq t\}$.

Any subset T of $\Lambda^{<\mathbb{N}}$ which contains all non-empty initial segments of its members will be called a B -tree. We define by transfinite induction the *derived B trees* of T . We let $\text{MAX}(T)$ denote the \preceq -maximal members of T and $T' = T \setminus \text{MAX}(T)$. We then define $T^0 = T$, $T^{\xi+1} = (T^\xi)'$, and if ξ is a limit ordinal, $T^\xi = \bigcap_{\gamma < \xi} T^\gamma$. We let $o(T)$ denote the smallest ordinal ξ such that $T^\xi = \emptyset$, provided such an ordinal exists. If no such ordinal exists, we write $o(T) = \infty$. We say T is *well-founded* if $o(T)$ is an ordinal, and T is *ill-founded* if $o(T) = \infty$. For convenience, we agree to the convention that if ξ is an ordinal $\xi < \infty$, and that $\omega\infty = \infty$.

Given a B -tree T and a Banach space Y , we let $T.Y = \{(\zeta_i, Z_i)_{i=1}^k : (\zeta_i)_{i=1}^k \in T, Z_i \in \text{codim}(Y)\}$, where $\text{codim}(Y)$ denotes the closed subspaces of Y having finite codimension in Y . We let \mathcal{C} denote the norm compact subsets of B_X and

$$T.X.\mathcal{C} = \{(\zeta_i, Z_i, C_i)_{i=1}^k : (\zeta_i)_{i=1}^k \in T, Z_i \in \text{codim}(X), C_i \in \mathcal{C}\}.$$

We note that $T.Y$ and $T.X.C$ are B -trees. Furthermore, for any ordinal γ , $(T.Y)^\gamma = T^\gamma.Y$ and $(T.X.C)^\gamma = T^\gamma.X.C$. In particular, $T.Y$ and $T.X.C$ have the same order as T .

Given a B -tree T , a Banach space Y , and a collection $(x_t)_{t \in T.Y} \subset Y$, we say $(x_t)_{t \in T.Y}$ is *normally weakly null* provided that for any $t = (\zeta_i, Z_i)_{i=1}^k \in T.Y$, $x_t \in Z_k$. Given another B -tree S and a function $\sigma : S.Y \rightarrow T.Y$, we say σ is a *pruning* provided that for every $s, s_1 \in S.Y$ with $s \prec s_1$, $\sigma(s) \prec \sigma(s_1)$, and if $s_1 = s^\wedge(\zeta, Z)$ and $\sigma(s_1) = t^\wedge(\mu, W)$ for some $t \in T.Y$, $W \leq Z$. If $\sigma : S.Y \rightarrow T.Y$ is a pruning and $\tau : \text{MAX}(S.Y) \rightarrow \text{MAX}(T.Y)$ is such that for every $s \in \text{MAX}(S.Y)$, $\sigma(s) \preceq \tau(s)$, we say the pair (σ, τ) is an *extended pruning*, and denote this by $(\sigma, \tau) : S.Y \rightarrow T.Y$.

For every $\xi \in \mathbb{N}$ and $n \in \mathbb{N}$, a B -tree $\Gamma_{\xi,n}$ was defined in [3] so that $o(\Gamma_{\xi,n}) = \omega^\xi n$. Furthermore, a function $\mathbb{P}_\xi : \Gamma_\xi \rightarrow [0, 1]$ was defined so that for every $t \in \text{MAX}(\Gamma_\xi)$, $\sum_{s \preceq t} \mathbb{P}_\xi(s) = 1$. Furthermore, $\Gamma_{\xi+1}$ is the disjoint union of $\Gamma_{\xi,n}$, $n \in \mathbb{N}$. For convenience, we define $\mathbb{P}_{\xi,n} : \Gamma_{\xi,n} \rightarrow [0, n]$ by $\mathbb{P}_{\xi,n}(s) = n\mathbb{P}_{\xi+1}(s)$. It follows from the definitions that $\Gamma_{\xi,1} = \Gamma_\xi$ and $\mathbb{P}_{\xi,1} = \mathbb{P}_\xi$. For every ξ and every $n \in \mathbb{N}$, there exist disjoint subsets $\Lambda_{\xi,n,1}, \dots, \Lambda_{\xi,n,n}$ of $\Gamma_{\xi,n}$ such that $\Gamma_{\xi,n} = \bigcup_{i=1}^n \Lambda_{\xi,n,i}$. It follows from the facts regarding $\mathbb{P}_{\xi+1}$ discussed in [3] that, with these definitions, for every ordinal ξ , every $n \in \mathbb{N}$, every $1 \leq i \leq n$, and every $t \in \text{MAX}(\Gamma_{\xi,n})$, $\sum_{\Lambda_{\xi,n,i} \ni s \preceq t} \mathbb{P}_{\xi,n}(s) = 1$. For any Banach space Y , we may define $\mathbb{P}_{\xi,n}$ on $\Gamma_{\xi,n}.Y$ and $\Gamma_{\xi,n}.X.C$ by letting

$$\mathbb{P}_{\xi,n}((\zeta_i, Z_i)_{i=1}^k) = \mathbb{P}_{\xi,n}((\zeta_i)_{i=1}^k)$$

and

$$\mathbb{P}_{\xi,n}((\zeta_i, Z_i, C_i)_{i=1}^k) = \mathbb{P}_{\xi,n}((\zeta_i)_{i=1}^k).$$

We say an extended pruning $(\sigma, \tau) : \Gamma_{\xi,n}.X \rightarrow \Gamma_{\xi,n}.X$ is *level preserving* provided that for every $1 \leq i \leq n$, $\sigma(\Lambda_{\xi,n,i}) \subset \Lambda_{\xi,n,i}$.

The following theorem collects results from Theorem 3.3, Propositions 3.2, 3.3, and Lemma 3.4 of [4].

Theorem 4. *Suppose ξ is an ordinal and n is a natural number.*

(i) *If $f : \Pi(\Gamma_{\xi,n}.X) \rightarrow \mathbb{R}$ is bounded and $\lambda \in \mathbb{R}$ is such that*

$$\lambda < \inf_{t \in \text{MAX}(\Gamma_{\xi,n}.X)} \sum_{s \preceq t} \mathbb{P}_{\xi,n}(s) f(s, t),$$

then there exist a level preserving extended pruning $(\sigma, \tau) : \Gamma_{\xi,n}.X \rightarrow \Gamma_{\xi,n}.X$ and real numbers b_1, \dots, b_n such that $\lambda < \sum_{i=1}^n b_i$ and for every $1 \leq i \leq n$ and every $\Lambda_{\xi,n,i} \ni s \preceq t \in \text{MAX}(\Gamma_{\xi,n}.X)$, $b_i \leq f(\sigma(s), \tau(t))$.

(ii) *If (M, d) is a compact metric space and $f : \Pi(\Gamma_{\xi,n}.X) \rightarrow M$ is any function, then for any $\delta > 0$, there exist $x_1, \dots, x_n \in M$ and a level preserving extended pruning $(\sigma, \tau) : \Gamma_{\xi,n}.X \rightarrow \Gamma_{\xi,n}.X$ such that for every $1 \leq i \leq n$ and every $\Lambda_{\xi,n,i} \ni s \preceq t \in \text{MAX}(\Gamma_{\xi,n}.X)$, $d(x_i, f(\sigma(s), \tau(t))) < \delta$.*

- (iii) If F is a finite set and $f : \text{MAX}(\Gamma_{\xi,n}.X) \rightarrow F$ is any function, there exists a level preserving extended pruning $(\sigma, \tau) : \Gamma_{\xi,n}.X \rightarrow \Gamma_{\xi,n}.X$ such that $f \circ \tau|_{\text{MAX}(\Gamma_{\xi,n}.X)}$ is constant.
- (iv) For any natural numbers $k_1 < \dots < k_r \leq n$, there exists an extended pruning $(\sigma, \tau) : \Gamma_{\xi,r}.X \rightarrow \Gamma_{\xi,n}.X$ such that for every $1 \leq i \leq r$, $\sigma(\Lambda_{\xi,n,i}) \subset \Lambda_{\xi,n,k_i}$.

2.2. The Szlenk index, Szlenk power type. Given a w^* -compact subset L of X^* and $\varepsilon > 0$, we let $s_\varepsilon(K)$ denote the set consisting of those $x^* \in L$ such that for every w^* -neighborhood V of x^* , $\text{diam}(L \cap V) > \varepsilon$. We define the transfinite derivations

$$s_\varepsilon^0(L) = L,$$

$$s_\varepsilon^{\xi+1}(L) = s_\varepsilon(s_\varepsilon^\xi(L)),$$

and if ξ is a limit ordinal,

$$s_\varepsilon^\xi(L) = \bigcap_{\zeta < \xi} s_\varepsilon^\zeta(L).$$

If there exists an ordinal ξ such that $s_\varepsilon^\xi(L) = \emptyset$, we let $Sz(L, \varepsilon)$ be the minimum such ordinal. Otherwise we write $Sz(L, \varepsilon) = \infty$. Since $s_\varepsilon^\xi(L)$ is w^* -compact, we deduce that $Sz(L, \varepsilon)$ cannot be a limit ordinal. We agree to the conventions that $\omega\infty = \infty$ and $\xi < \infty$ for any ordinal ξ . We let $Sz(L) = \sup_{\varepsilon > 0} Sz(L, \varepsilon)$. If $B : Z \rightarrow W$ is an operator, we let $Sz(B, \varepsilon) = Sz(B^*B_{W^*}, \varepsilon)$, $Sz(B) = Sz(B^*B_{W^*})$. If Z is a Banach space, $Sz(Z, \varepsilon) = Sz(I_Z, \varepsilon)$ and $Sz(Z) = Sz(I_Z)$.

We recall that a set $L \subset X^*$ is called w^* -fragmentable if for any $\varepsilon > 0$ and any w^* -compact, non-empty subset M of L , $s_\varepsilon(M) \subsetneq M$. This is equivalent to $Sz(L) < \infty$. We say an operator $B : Z \rightarrow W$ is Asplund if $B^*B_{W^*}$ is w^* -fragmentable, which happens if and only if $Sz(B) < \infty$. We say a Banach space Z is Asplund if I_Z is Asplund. These are not the original definitions of Asplund spaces and operators, but they are equivalent to the original definitions (see [1]).

If $Sz(K) \leq \omega^{\xi+1}$, then for any $\varepsilon > 0$, $Sz(K, \varepsilon) \leq \omega^\xi n$ for some $n \in \mathbb{N}$. We let $Sz_\xi(K, \varepsilon)$ be the smallest $n \in \mathbb{N}$ such that $Sz(K, \varepsilon) \leq \omega^\xi n$. We define the ξ -Szlenk power type $\mathbf{p}_\xi(K)$ of K by

$$\mathbf{p}_\xi(K) = \limsup_{\varepsilon \rightarrow 0^+} \frac{\log Sz_\xi(K, \varepsilon)}{|\log(\varepsilon)|}.$$

This value need not be finite. By convention, we let $\mathbf{p}_\xi(K) = \infty$ if $Sz(K) > \omega^{\xi+1}$. We let $\mathbf{p}_\xi(A) = \mathbf{p}_\xi(A^*B_{Y^*})$ and $\mathbf{p}_\xi(X) = \mathbf{p}_\xi(B_{X^*})$. The quantities $\mathbf{p}_\xi(X)$, $\mathbf{p}_\xi(A)$ are important for the renorming theorem of ξ -asymptotically uniformly smooth norms with power type modulus.

Given a w^* -compact subset L of X^* and $\varepsilon > 0$, we let $\mathcal{H}_\varepsilon^L$ denote the set of Cartesian products $\prod_{i=1}^n C_i$ such that $C_i \in \mathcal{C}$ for each $1 \leq i \leq n$ and such that there exist $(x_i)_{i=1}^n \in \prod_{i=1}^n C_i$ and $x^* \in K$ such that for each $1 \leq i \leq n$, $\text{Re } x^*(x_i) \geq \varepsilon$.

2.3. The Szlenk index of K_p . Recall that for $1 < p < \infty$, $L_p(X)$ denotes the space of equivalence classes of Bochner integrable functions $f : [0, 1] \rightarrow X$ such that $\int \|f\|^p < \infty$, where $[0, 1]$ is endowed with its Lebesgue measure. Recall also that if $1 < q < \infty$, $L_q(X^*)$ is isometrically included in $L_p(X)^*$ by the action

$$f \mapsto \int \langle g, f \rangle,$$

for $g \in L_q(X^*)$. We also recall that if $\varrho : X \rightarrow \mathbb{R}$ is any Lipschitz function, then for any $f \in L_p(X)$, $\varrho \circ f \in L_p$.

We note that the Szlenk index and the ξ Szlenk power type of K are unchanged by scaling K by a positive scalar or by replacing K with its balanced hull. Moreover, for a positive scalar c , $(cK)_p = cK_p$, which has the same Szlenk index and ξ -Szlenk power type as K_p . If $\mathbb{T}K$ is the balanced hull of K , $K_p \subset (\mathbb{T}K)_p$ and $Sz(K) = Sz(\mathbb{T}K)$ ([3, Lemma 2.2]) so that Theorem 1, Corollary 2, and Theorem 3 hold in general if they hold under the assumption that $K \subset B_{X^*}$ is balanced. Therefore we can and do assume throughout that $K \subset B_{X^*}$ and K is balanced.

Let $\varrho : X \rightarrow \mathbb{R}$ be given by $\varrho(x) = \max_{x^* \in K} \operatorname{Re} x^*(x)$. Since we have assumed K is balanced, $\varrho(x) = \max_{x^* \in K} |x^*(x)|$. It is easy to see that for any $1 < p < \infty$ and any $f \in L_p(X)$, $\|\varrho(f)\|_{L_p} = \max_{f^* \in K_p} \operatorname{Re} f^*(f)$. Combining this fact with [4, Corollary 2.4] and the proof of that corollary, we obtain the following.

Theorem 5. *Fix $1 < p, \alpha < \infty$.*

(i) *If for every B -tree T with $o(T) = \omega^{1+\xi}$ and every normally weakly null $(f_t)_{t \in T.L_p(X)} \subset B_{L_p(X)}$,*

$$\inf \{ \|\varrho(f)\|_{L_p} : t \in T.L_p(X), f \in \operatorname{co}(f_s : \emptyset \prec s \preceq t) \} = 0,$$

then $Sz(K_p) \leq \omega^{1+\xi}$.

(ii) *If there exists a constant C such that for every $n \in \mathbb{N}$, every B -tree T with $o(T) = \omega^{1+\xi}n$, and every normally weakly null collection $(f_t)_{t \in T.L_p(X)} \subset B_{L_p(X)}$,*

$$\inf \{ \|\varrho(f)\|_{L_p} : t \in T.L_p(X), f \in \operatorname{co}(f_s : \emptyset \prec s \preceq t) \} \leq Cn^{-1/\alpha},$$

then $\mathbf{p}_{1+\xi}(K_p) \leq \alpha$.

Proposition 6. *Suppose T is a non-empty B -tree. Suppose also that $(C_s)_{s \in T.X} \subset \mathcal{C}$ is fixed and for $s = (\zeta_i, Z_i)_{i=1}^k \in T.X$, let $\lambda(s) = Z_k \cap C_s$. Suppose that S is a non-empty, well-founded B -tree and $\theta : S.X \rightarrow T.X$ is a pruning. For $s \in S.X$, let $\mathfrak{s}(s) = \prod_{i=1}^{|s|} \lambda(\theta(s|_i))$. If $\varepsilon > 0$ is such that for every $t \in S.X$, $\mathfrak{s}(s) \in \mathcal{H}_\varepsilon^K \neq \emptyset$, then for any $0 < \delta < \varepsilon$, any $0 \leq \gamma < o(S)$, and any $s \in S^\gamma.X$, $\mathfrak{s}(s) \in \mathcal{H}_\varepsilon^{s_\delta^\gamma(K)} \neq \emptyset$. Moreover, for any $0 < \delta < \varepsilon$, $Sz(K, \delta) > o(S)$.*

Proof. We induct on γ . The base case is the hypothesis. Assume $\gamma + 1 < o(S)$ and the result holds for γ . Assume $s \in S^{\gamma+1}.X$, which means there exists ζ such that $s^\wedge(\zeta, Z) \in S^\gamma.X$ for all $Z \in \operatorname{codim}(X)$. Then for every $Z \in \operatorname{codim}(X)$, there exists $Z \geq W_Z \in \operatorname{codim}(X)$

such that $\mathfrak{s}(s^\wedge(\zeta, Z)) \subset \mathfrak{s}(s) \times B_{W_Z}$. From this and the inductive hypothesis, for every $Z \in \text{codim}(X)$, we fix $x_Z \in B_{W_Z}$, $(x_i^Z)_{i=1}^{|s|} \in \mathfrak{s}(s)$, and $x_Z^* \in s_\delta^\gamma(K)$ such that $\text{Re } x_Z^*(x_Z) \geq \varepsilon$ and $\text{Re } x_Z^*(x_i^Z) \geq \varepsilon$ for each $1 \leq i \leq |s|$. By compactness of $\mathfrak{s}(s) \times K$ with the product topology, where $\lambda(\theta(s|_i))$ has its norm topology and K has its w^* -topology,

$$\emptyset \neq \bigcap_{Z \in \text{codim}(X)} \overline{\{(x_1^Y, \dots, x_{|s|}^Y, x_Y^*) : Z \geq Y \in \text{codim}(X)\}} \subset \mathfrak{s}(s) \times K.$$

Fix $(x_1, \dots, x_{|s|}, x^*)$ lying in this intersection. Obviously $x^* \in s_\delta^\gamma(K)$. Moreover, for any w^* -neighborhood V of x^* , there exists $Z \in \text{codim}(X)$ such that $\ker(x^*) \subset Z$ and $x_Z^* \in V$, whence

$$\text{diam}(s_\delta^\gamma(K) \cap V) \geq \|x_Z^* - x^*\| \geq \text{Re } (x_Z^* - x^*)(x_Z) = \text{Re } x_Z^*(x_Z) \geq \varepsilon > \delta.$$

This implies $x^* \in s_\delta^{\gamma+1}(K)$. It is obvious that $\text{Re } x^*(x_i) \geq \varepsilon$ for all $1 \leq i \leq |s|$. This shows that $\mathfrak{s}(s) \in \mathcal{H}_\varepsilon^{s_\delta^{\gamma+1}(K)}$ and completes the successor case.

Finally, assume $\gamma < o(S)$ is a limit ordinal and the result holds for all ordinals less than γ . Fix $s \in S^\gamma.X$ and let $\mathfrak{s}(s) \times K$ be topologized as in the successor case. By the inductive hypothesis, for all $\beta < \gamma$, there exists $(x_1^\beta, \dots, x_{|s|}^\beta, x_\beta^*) \in \mathfrak{s}(s) \times K$ such that $x_\beta^* \in s_\varepsilon^\beta(K)$ and for all $1 \leq i \leq |s|$, $\text{Re } x_\beta^*(x_i^\beta) \geq \varepsilon$. By compactness of $(\prod_{i=1}^{|s|} \lambda(\theta(s|_i))) \times K$,

$$\bigcap_{\beta < \gamma} \overline{\{(x_1^\mu, \dots, x_{|s|}^\mu, x_\mu^*) : \mu \geq \beta\}} \neq \emptyset.$$

Clearly any $(x_1, \dots, x_{|s|}, x^*)$ lying in this intersection is such that $x^* \in s_\delta^\gamma(K)$ and for any $1 \leq i \leq |s|$, $\text{Re } x^*(x_i) \geq \varepsilon$. This shows that $\mathfrak{s}(s) \in \mathcal{H}_\varepsilon^{s_\delta^\gamma(K)}$ and completes the induction.

We have shown that for any $0 < \delta < \varepsilon$, $Sz(K, \delta) \geq o(S)$. If $o(S)$ is a limit ordinal, we deduce that $Sz(K, \delta) > o(S)$ since $Sz(K, \delta)$ cannot be a limit ordinal. If $o(S)$ is a successor, say $o(S) = \xi + 1$, then there exists a length 1 sequence $(\zeta) \in S^\xi$. For every $Z \in \text{codim}(X)$, $\mathfrak{s}((\zeta, Z)) = W_Z \cap C_{\theta((\zeta, Z))}$ for some $W \subset Z$. The first part of the proof yields that for each $Z \in \text{codim}(X)$, there exists $x_Z \in W_Z \cap C_{\theta((\zeta, Z))} \subset W_Z \cap B_X$ and some $x_Z^* \in s_\xi^\xi(K)$ such that $\text{Re } x_Z^*(x_Z) \geq \varepsilon$. Arguing as in the successor case, we deduce that any w^* -limit of a subnet of $(x_Z^*)_{Z \in \text{codim}(X)}$ lies in $s_\delta^{\xi+1}(K)$, whence $Sz(K, \delta) > \xi + 1 = o(S)$. □

2.4. Games. Suppose $T \subset \Lambda^{<\mathbb{N}}$ is a well-founded, non-empty B -tree and $\mathcal{E} \subset \text{MAX}(T.X.\mathcal{C})$ is some subset. We define the *game on $T.X.\mathcal{C}$ with target set \mathcal{E}* . Player I first chooses $(\zeta_1, Z_1) \in \Lambda \times \text{codim}(X)$ such that $(\zeta) \in T$ and Player II then chooses $C_1 \in \mathcal{C}$. Assuming $(\zeta_i, Z_i)_{i=1}^n \in T.X$ and $C_1, \dots, C_n \in \mathcal{C}$ have been chosen, the game terminates if $(\zeta_i, Z_i)_{i=1}^n \in \text{MAX}(T.X)$. Otherwise Player I chooses $(\zeta_{n+1}, Z_{n+1}) \in \Lambda \times \text{codim}(X)$ such that $(\zeta_i)_{i=1}^{n+1} \in T$ and Player II chooses $C_{n+1} \in \mathcal{C}$. Since T is well-founded, this game must terminate after finitely many steps. Suppose that the resulting choices are $(\zeta_i, Z_i)_{i=1}^n$ and $C_1, \dots, C_n \in \mathcal{C}$. We say that Player II wins if $(\zeta_i, Z_i, C_i)_{i=1}^n \in \mathcal{E}$, and Player I wins otherwise.

A *strategy for Player I* for the game on $T.X.C$ with target set \mathcal{E} is a function $\psi : T'.X.C \cup \{\emptyset\} \rightarrow \Lambda \times \text{codim}(X)$ such that if $\psi((\zeta_i, Z_i, C_i)_{i=1}^{n-1}) = (\zeta_n, Z_n)$, $(\zeta_i)_{i=1}^n \in T$. We say ψ is a *winning strategy for Player I* provided that for any sequence $(\zeta_i, Z_i, C_i)_{i=1}^n \in \text{MAX}(T.X.C)$ such that $(\zeta_i, Z_i) = \psi((\zeta_j, Z_j, C_j)_{j=1}^{i-1})$ for every $1 \leq i \leq n$, $(\zeta_i, Z_i, C_i)_{i=1}^n \notin \mathcal{E}$.

A *strategy for Player II* for the game on $T.X.C$ with target set \mathcal{E} is a function ψ defined on the set

$$\{((\zeta_i, Z_i, C_i)_{i=1}^{n-1}, (\zeta_n, Z_n)) : (\zeta_i, Z_i, C_i)_{i=1}^{n-1} \in \{\emptyset\} \cup T.X.C, (\zeta_n, Z_n) \in \Lambda \times \text{codim}(X), (\zeta_i)_{i=1}^n \in T\}$$

and taking values in C . We say ψ is a *winning strategy for Player II* provided that for any sequence $(\zeta_i, Z_i, C_i)_{i=1}^n \in \text{MAX}(T.X.C)$ such that $C_i = \psi((\zeta_j, Z_j, C_j)_{j=1}^{i-1}, (\zeta_i, Z_i))$ for all $1 \leq i \leq n$, $(\zeta_i, Z_i, C_i)_{i=1}^n \in \mathcal{E}$.

Proposition 7. [5, Proposition 3.1] *For any non-empty, well-founded B-tree T and any $\mathcal{E} \subset T.X.C$, either Player I or Player II has a winning strategy for the game on $T.X.C$ with target set \mathcal{E} .*

Proposition 8. *Suppose that Player II has a winning strategy for a game on $T.X.C$ with target set \mathcal{E} . Then there exists $(C_s)_{s \in T.X} \subset C$ such that for every $t = (\zeta_i, Z_i)_{i=1}^k \in \text{MAX}(T.X)$, $(\zeta_i, Z_i, C_{t|_i})_{i=1}^k \in \mathcal{E}$.*

Proof. Fix a winning strategy ψ for Player II in the game. We define C_s by induction on $|s|$. We let $C_{(\zeta, Z)} = \psi(\emptyset, (\zeta, Z))$. If $|s| = k + 1$, $C_{s|_i}$ has been defined for every $1 \leq i \leq k$, and $s = s|_k \widehat{\ }_k (\zeta, Z)$, we let $C_s = \psi(s|_k, (\zeta, Z))$. □

For the next proposition, if $h \in L_p(X)$ is a simple function, we let \bar{h} be the function in $L_p(X)$ such that $\bar{h}(\varpi) = 0$ if $h(\varpi) = 0$ and $\bar{h}(\varpi) = h(\varpi)/\|h(\varpi)\|$ otherwise.

Proposition 9. *Let ξ be an ordinal, n a natural number, and let T be a B-tree with $o(T) \geq \omega^{1+\xi}n$. If ψ is a strategy for Player I for some game on $\Gamma_{\xi, n}.X.C$, then for any $1 < p < \infty$, any $\delta > 0$, and any normally weakly null $(f_t)_{t \in T.L_p(X)} \subset B_{L_p(X)}$, there exist $s = (\zeta_i, Z_i)_{i=1}^k \in \text{MAX}(\Gamma_{\xi, n}.X)$, $\emptyset = t_0 \prec t_1 \prec \dots \prec t_k \in T.L_p(X)$, $g_i \in \text{co}(f_u : t_{i-1} \prec u \preceq t_i)$, $h_i \in B_{L_p(X)}$, and $C_i \in C$ such that for every $1 \leq i \leq k$,*

- (i) h_i is simple,
- (ii) $\text{range}(\bar{h}_i) = C_i \subset B_{Z_i}$,
- (iii) $\|g_i - h_i\|_{L_p(X)} < \delta$,
- (iv) $(\zeta_i, Z_i) = \psi((\zeta_j, Z_j, C_j)_{j=1}^{i-1})$.

Remark 10. For a B-tree S on Λ and $s \in S$, we let $S(s)$ denote those non-empty sequences $u \in \Lambda^{<\mathbb{N}}$ such that $s \widehat{\ } u \in S$. An easy induction argument yields that for any ordinals ξ, ζ , $S^\xi(s) = (S(s))^\xi$ for any ordinal ξ . From this it follows that $s \in S^\xi$ if and only if $o(S(s)) \geq \xi$.

Furthermore, another easy induction yields that if $(S^\xi)^\zeta = S^{\xi+\zeta}$, from which it follows that if $o(S) \geq \xi + \zeta$, $o(S^\xi) \geq \zeta$. Therefore if $s \in S^{\xi+\omega}$, $o(S^\xi(s)) \geq \omega$.

Proof of Proposition 9. We first note that if $Z \in \text{codim}(X)$, $L_p(X)/L_p(Z)$ is either the zero vector space or isomorphic to L_p , and therefore has Szlenk index not exceeding ω . As explained in [5], this means that for any B -tree T with $o(T) \geq \omega$, any $\delta > 0$, and any normally weakly null $(f_t)_{t \in T.L_p(X)} \subset B_{L_p(X)}$, there exist $t \in T.L_p(X)$, $g \in \text{co}(f_s : \emptyset \prec s \preceq t)$, and $h \in B_{L_p(Z)}$ such that $\|g - h\|_{L_p(X)} < \delta$. Moreover, by the density of simple functions, we may assume this h is simple.

Let ψ be a strategy for Player I for a game on $\Gamma_{\xi,n}.X.C$. Let T be a B -tree with $o(T) = \omega^{1+\xi}n$ and define $\gamma : \Gamma_{\xi,n}.X \cup \{\emptyset\} \rightarrow [0, \omega^\xi n]$ by letting $\gamma(t) = \max\{\mu \leq \omega^\xi n : t \in (\Gamma_{\xi,n}.X)^\mu\}$ for $t \in \Gamma_{\xi,n}.X$ and $\gamma(\emptyset) = \omega^\xi n$. Let $s_0 = t_0 = \emptyset$. Now assume that for some $k \in \mathbb{N}$ and all $1 \leq i < k$, $s_i \in \Gamma_{\xi,n}.X$, $\zeta_i \in [0, \omega^\xi n]$, $Z_i \in \text{codim}(X)$, $t_i \in T.L_p(X)$, $g_i, h_i \in B_{L_p(X)}$, and $C_i \in \mathcal{C}$ have been chosen such that for all $1 \leq i < k$,

- (i) h_i is simple,
- (ii) $s_i = (\zeta_j, Z_j)_{j=1}^i$,
- (iii) $t_0 \prec t_1 \prec \dots \prec t_{k-1}$,
- (iv) $t_i \in (T.L_p(X))^{\omega^{\gamma(s_i)}}$,
- (v) $(\zeta_i, Z_i) = \psi((\zeta_j, Z_j, C_j)_{j=1}^{i-1})$,
- (vi) $g_i \in \text{co}(f_u : t_{i-1} \prec u \preceq t_i)$,
- (vii) $\|g_i - h_i\|_{L_p(X)} < \delta$,
- (viii) $\text{range}(\bar{h}_i) = C_i \subset B_{Z_i}$.

If s_{k-1} is maximal in $\Gamma_{\xi,n}.X$, we let $s = s_{k-1}$, and one easily checks that the conclusions are satisfied. Otherwise let $(\zeta_k, Z_k) = \psi((\zeta_j, Z_j, C_j)_{j=1}^{k-1})$ and $s_k = \widehat{s_{k-1}}(\zeta_k, Z_k)$. Let u_{k-1} be the sequence of first members of the pairs of t_{k-1} and let U denote the proper extensions of u_{k-1} in $T^{\omega^{\gamma(s_k)}}$. Then $(f_{\widehat{t_{k-1}u}})_{u \in U.L_p(X)} \subset B_{L_p(X)}$ is normally weakly null and $o(U) \geq \omega$ by the remark preceding the proof, so that the previous paragraph yields the existence of some $u' \in U.L_p(X)$, $g_k \in \text{co}(f_u : t_{k-1} \prec u \preceq \widehat{t_{k-1}u'})$, and some simple function $h_k \in B_{L_p(Z_k)}$ such that $\|g_k - h_k\|_{L_p(X)} < \delta$. Let $t_k = \widehat{t_{k-1}u'}$. In order to apply the remark before the proof, we note that since $s_{k-1} \prec s_k$, $\gamma(s_{k-1}) \geq \gamma(s_k) + 1$. Since

$$\omega^{\gamma(s_{k-1})} \geq \omega^{\gamma(s_k) + 1} = \omega^{\gamma(s_k)} + \omega,$$

the remark preceding the proof applies. Note that $C_k := \text{range}(\bar{h}_k) \subset B_{Z_k}$. This completes the recursive construction. Since $\Gamma_{\xi,n}.X$ is well-founded, eventually this process terminates. The resulting $s = (\zeta_i, Z_i)_{i=1}^k \in \text{MAX}(\Gamma_{\xi,n}.X)$ clearly satisfies the conclusions. \square

3. DEFINITION OF AN ASSOCIATED SPACE AND TWO GAMES

3.1. The associated space and its properties. If E is a vector space with seminorm $\|\cdot\|$, we say a sequence $(e_i)_{i=1}^n$ in E is *1-unconditional* provided that for any scalars $(a_i)_{i=1}^n$

and any $(\varepsilon_i)_{i=1}^n \in \{\pm 1\}^n$, $\|\sum_{i=1}^n \varepsilon_i a_i e_i\| = \|\sum_{i=1}^n a_i e_i\|$. Recall that for $1 < p < \infty$, a vector space E with seminorm $\|\cdot\|$ which is spanned by the 1-unconditional basis $(e_i)_{i=1}^n$ is called *p-concave* provided there exists a constant C such that for any $(f_i)_{i=1}^n \subset L_p$,

$$\left\| \sum_{i=1}^n f_i e_i \right\|_{L_p(E)} \leq C \left\| \sum_{i=1}^n \|f_i\|_{L_p} e_i \right\|_E.$$

The smallest such constant C is denoted by $M_{(p)}(E)$.

Given $x \in \text{span}(e_i : 1 \leq i \leq n)$, where $(e_i)_{i=1}^n$ is a Hamel basis for the seminormed space E , we write $x = \sum_{i=1}^n a_i e_i$ and $\text{supp}(x) = \{i \leq n : a_i \neq 0\}$. We say the vectors $x_1, \dots, x_n \in \text{span}(e_i : 1 \leq i \leq n)$ are *disjointly supported* if the sets $\text{supp}(x_1), \dots, \text{supp}(x_n)$ are pairwise disjoint.

For $1 < \beta < \infty$, we say that an unconditional Hamel basis $(e_i)_{i=1}^n$ for a seminormed space E satisfies a 1-lower ℓ_β estimate provided that for any $m \in \mathbb{N}$ and any disjointly supported elements $(x_i)_{i=1}^m \subset E$,

$$\left(\sum_{i=1}^m \|x_i\|^\beta \right)^{1/\beta} \leq \left\| \sum_{i=1}^m x_i \right\|.$$

Theorem 11. [7, Theorem 1.f.7] *Fix $1 < \beta < p < \infty$. There exists a constant $C' = C'(\beta, p)$ such that if $(e_i)_{i=1}^n$ is a 1-unconditional basis for the seminormed space E which satisfies a 1-lower ℓ_β estimate, then E is p -concave and $M_{(p)}(E) \leq C'$.*

For the remainder of this section, T is a fixed, non-empty B -tree.

For a non-empty set J , we let $c_{00}(J)$ be the span of the canonical Hamel basis $(e_j)_{j \in J}$ in the space of scalar-valued functions on J , where e_j is the indicator of the singleton $\{j\}$. We let e_j^* denote the coordinate functional to e_j . Given $x \in c_{00}(J)$, we may write $x = \sum_{j \in J} a_j e_j$. Then we define $|x|$ to be $\sum_{j \in J} |a_j| e_j$. A *suppression projection* is an operator P from $\text{span}(e_j^* : j \in J)$ into itself such that there exists a subset F of J such that $P \sum_{j \in J} a_j e_j^* = \sum_{j \in F} a_j e_j^*$.

For $0 < \phi < \theta < 1$, let

$$N_{\theta, \phi, T} = \{0\} \cup \left\{ \theta \sum_{i=1}^k e_{t|j_i}^* : t = (\zeta_i, Z_i, C_i)_{i=1}^{|t|} \in T.X.C, 1 \leq j_1 < \dots < j_k \leq |t|, \right. \\ \left. \prod_{i=1}^k Z_{j_i} \cap C_{j_i} \in \mathcal{H}_\phi^K \right\} \subset \text{span}(e_t^* : t \in T.X.C).$$

For $0 < \phi < \theta < 1$ and $1 < \alpha < \infty$, let

$$M_{\theta, \phi, \alpha, T} = \left\{ \sum_{i=1}^k a_i g_i : g_i \in \cup_{n=1}^\infty N_{\theta^n, \phi^n, T}, a_i \geq 0, \sum_{i=1}^k a_i^\alpha \leq 1, \text{supp}(g_i) \text{ are pairwise disjoint} \right\}.$$

Note that the set $M_{\theta, \phi, \alpha, T}$ is closed under suppression projections.

We define the seminorm $\|\cdot\|_{\theta, \phi, \alpha, T}$ on $c_{00}(T.X.C)$ by

$$\|x\|_{\theta, \phi, \alpha, T} = \sup\{f(|x|) : f \in M_{\theta, \phi, \alpha, T}\}.$$

Claim 12. Fix $1 < \alpha < \infty$ and $0 < \phi < \theta < 1$. For any $t \in T.X.C$, $(e_{t|i})_{i=1}^{|t|}$ is 1-unconditional and satisfies a 1-lower ℓ_β estimate in its span, where $1/\alpha + 1/\beta = 1$.

Proof. Note that 1-unconditionality is obvious. Fix $x_1, \dots, x_n \in \text{span}(e_{t|i} : 1 \leq i \leq |t|)$ with disjoint supports. That is, there exist pairwise disjoint subsets S_1, \dots, S_n of $\{1, \dots, |t|\}$ such that $x_i \in \text{span}(e_{t|j} : j \in S_i)$. Then there exist $g_1, \dots, g_n \in M_{\theta, \phi, \alpha, T}$ such that for each $1 \leq i \leq n$, $g_i(|x_i|) = \|x_i\|_{\theta, \phi, \alpha, T}$. Since $M_{\theta, \phi, \alpha, T}$ is closed under suppression projections, we may assume that $\text{supp}(g_i) \subset S_i$ for each $1 \leq i \leq n$. Then if $(a_i)_{i=1}^n$ are such that $\sum_{i=1}^n a_i^\alpha = 1$, $a_i \geq 0$, and $\sum_{i=1}^n a_i \|x_i\|_{\theta, \phi, \alpha, T} = (\sum_{i=1}^n \|x_i\|_{\theta, \phi, \alpha, T}^\beta)^{1/\beta}$, $g := \sum_{i=1}^n a_i g_i \in M_{\theta, \phi, \alpha, T}$ and

$$\left\| \sum_{i=1}^n x_i \right\|_{\theta, \phi, \alpha, T} \geq g\left(\left| \sum_{i=1}^n x_i \right|\right) = \sum_{i=1}^n a_i g_i(|x_i|) = \left(\sum_{i=1}^n \|x_i\|_{\theta, \phi, \alpha, T}^\beta\right)^{1/\beta}.$$

□

Claim 13. Fix $1 < \alpha < \infty$ and $0 < \phi < \theta < 1$. For any $t = (\zeta_i, Z_i, C_i)_{i=1}^k \in T.X.C$, any sequence $(x_i)_{i=1}^k \in \prod_{i=1}^k Z_i \cap C_i$, and any sequence $(a_i)_{i=1}^k$ of non-negative scalars,

$$\varrho\left(\sum_{i=1}^k a_i x_i\right) \leq \frac{1}{\theta - \phi} \left\| \sum_{i=1}^k a_i e_{t|i} \right\|_{\theta, \phi, \alpha, T}.$$

Proof. We recall that if $C \in \mathcal{C}$, $C \subset B_X$ by the definition of \mathcal{C} . With t , $(x_i)_{i=1}^k \in \prod_{i=1}^k Z_i \cap C_i$, and $(a_i)_{i=1}^k$ as in the statement, fix $x^* \in K$ such that $\text{Re } x^*(\sum_{i=1}^k a_i x_i) = \varrho(\sum_{i=1}^k a_i x_i)$. For all $j \in \mathbb{N}$, let $B_j = \{i \leq k : \text{Re } x^*(x_i) \in (\phi^j, \phi^{j-1}]\}$. Note that for every $j \in \mathbb{N}$, $\theta^j \sum_{i \in B_j} e_{t|i}^* \in N_{\theta^j, \phi^j, T}$, so

$$\phi^{j-1} \sum_{i \in B_j} a_i = \phi^{-1} (\phi/\theta)^j (\theta^j \sum_{i \in B_j} e_{t|i}^*) \left(\sum_{i=1}^k a_i e_{t|i} \right) \leq \phi^{-1} (\phi/\theta)^j \left\| \sum_{i=1}^k a_i e_{t|i} \right\|_{\theta, \phi, \alpha, T}.$$

Then

$$\begin{aligned} \varrho\left(\sum_{i=1}^k a_i x_i\right) &\leq \sum_{j=1}^{\infty} \sum_{i \in B_j} a_i \text{Re } x^*(x_i) \leq \sum_{j=1}^{\infty} \phi^{j-1} \sum_{i \in B_j} a_i \leq \sum_{j=1}^{\infty} \phi^{-1} (\phi/\theta)^j \left\| \sum_{i=1}^k a_i e_{t|i} \right\|_{\theta, \phi, \alpha, T} \\ &= \frac{1}{\theta - \phi} \left\| \sum_{i=1}^k a_i e_{t|i} \right\|_{\theta, \phi, \alpha, T}. \end{aligned}$$

□

Corollary 14. Fix $1 < p, \alpha, \beta < \infty$ with $1/\alpha + 1/\beta = 1$ and $\beta < p$. Let $C' = C'(\beta, p)$ be the constant from Theorem 11. Suppose that ξ is an ordinal, n is a natural number, $\varepsilon > 0$, and $0 < \phi < \theta < 1$ are such that Player I has a winning strategy in the game with target set

$$\left\{ t \in \text{MAX}(\Gamma_{\xi, n}.X.C) : \left\| \sum_{s \leq t} \mathbb{P}_{\xi, n}(s) e_s \right\|_{\theta, \phi, \alpha, \Gamma_{\xi, n}} > \varepsilon \right\}.$$

Then for any B -tree T with $o(T) \geq \omega^{1+\varepsilon}n$ and any normally weakly null $(f_t)_{t \in T.L_p(X)} \subset B_{L_p(X)}$,

$$\inf \{ \|\varrho(f)\|_{L_p} : t \in T.L_p(X), f \in \text{co}(f_s : \emptyset \prec s \preceq t) \} \leq \frac{C'\varepsilon}{n(\theta - \phi)}.$$

Proof. Recall for the proof that for a simple function $h \in L_p(X)$, \bar{h} is the function in $L_p(X)$ such that $\bar{h}(\varpi) = 0$ if $h(\varpi) = 0$ and $\bar{h}(\varpi) = h(\varpi)/\|h(\varpi)\|$ otherwise.

Fix a winning strategy ψ for Player I in the game with the indicated target set. Fix $\delta > 0$. By Proposition 9, there exist $s = (\zeta_i, Z_i)_{i=1}^k \in \text{MAX}(\Gamma_{\xi,n}.X)$, $\emptyset = t_0 \prec \dots \prec t_k$, $g_i \in \text{co}(f_u : t_{i-1} \prec u \preceq t_i)$, simple functions $h_i \in B_{L_p(X)}$, and $C_i \in \mathcal{C}$ such that $\|g_i - h_i\|_{L_p(X)} < \delta$, $\text{range}(\bar{h}_i) = C_i \subset B_{Z_i}$, and $(\zeta_i, Z_i) = \psi((\zeta_j, Z_j, C_j)_{j=1}^{i-1})$. This means that for any $\varpi \in [0, 1]$, $(\bar{h}_i(\varpi))_{i=1}^k \in \prod_{i=1}^k Z_i \cap C_i$, whence by Claim 13, for any non-negative scalars $(a_i)_{i=1}^k$,

$$\varrho\left(\sum_{i=1}^k a_i h_i(\varpi)\right) = \varrho\left(\sum_{i=1}^k a_i \|h_i(\varpi)\| \bar{h}_i(\varpi)\right) \leq \frac{1}{\theta - \phi} \left\| \sum_{i=1}^k a_i \|h_i(\varpi)\| e_{s|i} \right\|_{\theta, \phi, \alpha, \Gamma_{\xi,n}}.$$

Since by Claim 13 $(e_u)_{u \preceq s}$ satisfies a lower ℓ_β estimate in its span, we deduce that

$$\begin{aligned} \left\| \varrho\left(\sum_{i=1}^k n^{-1} \mathbb{P}_{\xi,n}(s|i) h_i\right) \right\|_{L_p} &= \left(\int_0^1 \left| \varrho\left(\sum_{i=1}^k n^{-1} \mathbb{P}_{\xi,n}(s|i) \|h_i(\varpi)\| \bar{h}_i(\varpi)\right) \right|^p d\varpi \right)^{1/p} \\ &\leq \frac{1}{n(\theta - \phi)} \left(\int_0^1 \left\| \sum_{u \preceq s} \mathbb{P}_{\xi,n}(u) \|h_{|u|}(\varpi)\| e_u \right\|_{\theta, \phi, \alpha, \Gamma_{\xi,n}}^p d\varpi \right)^{1/p} \\ &\leq \frac{C'}{n(\theta - \phi)} \left\| \sum_{u \preceq s} \mathbb{P}_{\xi,n}(u) \|h_{|u|}\|_{L_p(X)} e_u \right\|_{\theta, \phi, \alpha, \Gamma_{\xi,n}} \\ &\leq \frac{C'\varepsilon}{n(\theta - \phi)}. \end{aligned}$$

Here we have used 1-unconditionality, $\|h_i\|_{L_p(X)} \leq 1$ for each $1 \leq i \leq k$, and the fact that since ψ is a winning strategy for Player I,

$$\left\| \sum_{u \preceq s} \mathbb{P}_{\xi,n}(u) \|h_{|u|}\|_{L_p(X)} e_u \right\|_{\theta, \phi, \alpha, \Gamma_{\xi,n}} \leq \left\| \sum_{u \preceq s} \mathbb{P}_{\xi,n}(u) e_u \right\|_{\theta, \phi, \alpha, \Gamma_{\xi,n}} \leq \varepsilon.$$

Let $g = n^{-1} \sum_{u \preceq s} \mathbb{P}_{\xi,n}(u) g_i \in \text{co}(f_u : u \preceq t_k)$ and $h = n^{-1} \sum_{u \preceq s} \mathbb{P}_{\xi,n}(u) h_i$. Since ϱ is 1-Lipschitz, it follows that $\|\varrho(g) - \varrho(h)\|_{L_p} \leq \|g - h\|_{L_p(X)} < \delta$, so that

$$\|\varrho(g)\|_{L_p} \leq \delta + \|\varrho(h)\|_{L_p} \leq \delta + \frac{C'\varepsilon}{n(\theta - \phi)}.$$

Since $\delta > 0$ was arbitrary, we are done. □

3.2. Particular games on $\Gamma_{\xi,n}.X.C$. The statement of Proposition 6 is notationally cumbersome. We isolate the following result as a way of using Proposition 6.

Lemma 15. *Fix $0 < \phi < \theta < 1$. Suppose that ξ is an ordinal, m, n are natural numbers, $(C_s)_{s \in \Gamma_{\xi,n}.X} \subset \mathcal{C}$, and $(\sigma, \tau) : \Gamma_{\xi,m}.X \rightarrow \Gamma_{\xi,n}.X$ is an extended pruning. For $t = (\zeta_i, Z_i)_{i=1}^k \in \Gamma_{\xi,n}.X$, let $r(t) = (\zeta_i, Z_i, C_{t|_i})_{i=1}^k$. If $\nu \in \mathbb{N}$ is such that for every $s \in \text{MAX}(\Gamma_{\xi,m}.X)$, there exists a functional $h_s \in \cup_{l=1}^\nu N_{\theta^l, \phi^l, \Gamma_{\xi,n}}$ such that $\cup_{t \preceq s} r(\sigma(t)) \subset \text{supp}(h_s)$, then $Sz(K, \phi^\nu/2) > \omega^\xi m$.*

Proof. For $s = (\zeta_i, Z_i)_{i=1}^k \in \Gamma_{\xi,n}.X$, let $\lambda(s) = Z_k \cap C_s$. For $s \in \Gamma_{\xi,m}.X$, let $\mathfrak{s}(s) = \prod_{i=1}^{|s|} \lambda(\sigma(s|_i))$.

Fix $s \in \text{MAX}(\Gamma_{\xi,m}.X)$ and let $h_s \in \cup_{l=1}^\nu N_{\theta^l, \phi^l, \Gamma_{\xi,n}}$ be as in the statement of the lemma and fix $1 \leq l \leq \nu$ such that $h_s \in N_{\theta^l, \phi^l, \Gamma_{\xi,n}}$. We will prove that $\mathfrak{s}(s) \in \mathcal{H}_{\phi^\nu}^K$. Since for any $1 \leq m \leq k$ and any $C'_1, \dots, C'_k \in \mathcal{C}$ such that $\prod_{i=1}^m C'_i \in \mathcal{H}_{\phi^\nu}^K$, $\prod_{i=1}^m C'_i \in \mathcal{H}_{\phi^\nu}^K$, this will show that for any non-empty initial segment s_1 of s , $\mathfrak{s}(s_1) \in \mathcal{H}_{\phi^\nu}^K$. From here, an appeal to Proposition 6 will finish the proof.

Fix $u = (\mu_i, W_i, C_i)_{i=1}^{|u|} \in \Gamma_{\xi,n}.X.C$ and $1 \leq j_1 < \dots < j_\mu \leq |u|$ such that $h_s = \theta^l \sum_{i=1}^\mu e_{u|_{j_i}}^*$ and $\prod_{i=1}^\mu W_{j_i} \cap C_{j_i} \in \mathcal{H}_{\phi^l}^K$. Let $\tau(s) = t = (\zeta_i, Z_i)_{i=1}^\eta$. For each $1 \leq i \leq |s|$, let $l_i = |\sigma(s|_i)|$. Note that for all $1 \leq i \leq |s|$, $r(\sigma(s|_i)) = (\zeta_j, Z_j, C_{t|_j})_{j=1}^{l_i}$ and $\mathfrak{s}(s) = \prod_{j=1}^{|s|} Z_{l_j} \cap C_{t|_{l_j}}$. By hypothesis,

$$\begin{aligned} \{(\zeta_j, Z_j, C_{t|_j})_{j=1}^{l_i} : 1 \leq i \leq |s|\} &= \{r(\sigma(s|_i)) : 1 \leq i \leq |s|\} \\ &\subset \text{supp}(h_s) = \{(\mu_j, W_j, C_j)_{j=1}^{j_i} : 1 \leq i \leq \mu\}. \end{aligned}$$

From this it follows that there exist $m_1 < \dots < m_{|s|}$ such that for every $1 \leq i \leq |s|$, $r(\sigma(s|_i)) = u|_{j_{m_i}}$. Choose $(x_i)_{i=1}^\mu \in \prod_{i=1}^\mu W_{j_i} \cap C_{j_i}$ such that there exists $x^* \in K$ so that $\text{Re } x^*(x_i) \geq \phi^l$ for each $1 \leq i \leq \mu$, which exists because $\prod_{i=1}^\mu W_{j_i} \cap C_{j_i} \in \mathcal{H}_{\phi^l}^K$. Since $Z_{l_i} = W_{j_{m_i}}$ and $C_{t|_{l_i}} = C_{j_{m_i}}$, $(x_{m_i})_{i=1}^{|s|} \in \prod_{i=1}^{|s|} Z_{l_i} \cap C_{t|_{l_i}}$, which shows that $\mathfrak{s}(s) \in \mathcal{H}_{\phi^l}^K$. Since $l \leq \nu$, $\mathcal{H}_{\phi^l}^K \subset \mathcal{H}_{\phi^\nu}^K$, so that $\mathfrak{s}(s) \in \mathcal{H}_{\phi^\nu}^K$. \square

Lemma 16. *Fix $1 < \alpha < \infty$ and $0 < \phi < \theta < 1$. If $Sz(K) \leq \omega^\xi$, then for any $\varepsilon > 0$, Player I has a winning strategy in the game with target set*

$$\left\{ t \in \text{MAX}(\Gamma_\xi.X.C) : \left\| \sum_{s \preceq t} \mathbb{P}_\xi(s) e_s \right\|_{\theta, \phi, \alpha, \Gamma_\xi} > \varepsilon \right\}.$$

Proof. Suppose not. Then by Proposition 8, there exist $\varepsilon > 0$ and $(C_s)_{s \in \Gamma_\xi.X} \subset \mathcal{C}$ such that

$$\varepsilon < \inf \left\{ \left\| \sum_{s \preceq t} \mathbb{P}_\xi(s) e_{r(s)} \right\|_{\theta, \phi, \alpha, \Gamma_\xi} : t \in \text{MAX}(\Gamma_\xi.X) \right\}.$$

For $s = (\zeta_i, Z_i)_{i=1}^k \in \Gamma_\xi.X$, let $r(s) = (\zeta_i, Z_i, C_{s|_i})_{i=1}^k$. For every $t \in \text{MAX}(\Gamma_\xi.X)$, fix $f_t \in M_{\theta, \phi, \alpha, \Gamma_\xi}$ such that $\text{supp}(f_t) \subset [\preceq r(t)]$ and $f_t(\sum_{s \preceq t} \mathbb{P}_\xi(s) e_{r(s)}) = \left\| \sum_{s \preceq t} \mathbb{P}_\xi(s) e_{r(s)} \right\|_{\theta, \phi, \alpha, \Gamma_\xi}$.

Define $F : \Pi(\Gamma_\xi.X) \rightarrow \mathbb{R}$ by letting $F(s, t) = f_t(e_{r(s)})$. By Theorem 4, there exists an extended pruning $(\sigma, \tau) : \Gamma_\xi.X \rightarrow \Gamma_\xi.X$ such that

$$\varepsilon < \inf_{(s,t) \in \Pi(\Gamma_\xi.X)} F(\sigma(s), \tau(t)).$$

Fix $\nu \in \mathbb{N}$ such that $\varepsilon > \theta^\nu$ and for each $t \in \text{MAX}(\Gamma_\xi.X)$, write $f_{\tau(t)} = \sum_{i=1}^{k_t} a_{i,t} g_{i,t}$ where $a_{i,t} \geq 0$, $\sum_{i=1}^{k_t} a_{i,t}^\alpha \leq 1$, and $g_{i,t} \in \cup_{n=1}^\infty N_{\theta^n, \phi^n, \Gamma_\xi}$ have pairwise disjoint supports. For each $t \in \text{MAX}(\Gamma_\xi.X)$, let

$$R_t = \{i \leq k_t : a_{i,t} \geq \varepsilon\}.$$

Since $\sum_{i=1}^{k_t} a_{i,t}^\alpha \leq 1$, $|R_t| \leq \lfloor 1/\varepsilon^\alpha \rfloor =: k_0$. Note that since $\varepsilon < f_{\tau(t)}(e_{r(\sigma(s))})$ for any $\emptyset \prec s \preceq t$, $r(\sigma(s)) \in \cup_{i \in R_t} \text{supp}(g_{i,t})$. We write $\sum_{i \in R_t} a_{i,t} g_{i,t} = \sum_{i=1}^{l_t} b_{i,t} h_{i,t}$ where $l_t \leq k_0$, $(b_{i,t})_{i=1}^{l_t}$ is an enumeration of $(a_{i,t})_{i \in R_t}$, and $(h_{i,t})_{i=1}^{l_t}$ is the corresponding enumeration of $(g_{i,t})_{i \in R_t}$. Define $\kappa : \Pi(\Gamma_\xi.X) \rightarrow \{1, \dots, k_0\}$ by letting $\kappa(\sigma, \tau)$ be the unique $i \leq l_t$ such that $r(\sigma(s)) \in \text{supp}(h_{i,t})$. By Theorem 4(ii), there exists an extended pruning $(\sigma', \tau') : \Gamma_\xi.X \rightarrow \Gamma_\xi.X$ and $1 \leq l \leq k_0$ such that $\kappa(\sigma'(s), \tau'(t)) = l$ for all $(s, t) \in \Pi(\Gamma_\xi.X)$. We now note that for any $s \in \text{MAX}(\Gamma_\xi.X)$, $h_{l, \tau'(s)} \in \cup_{i=1}^\nu N_{\theta^i, \phi^i, \Gamma_\xi}$ is such that

$$\{r(\sigma \circ \sigma'(u)) : \emptyset \prec u \preceq s\} \subset \text{supp}(h_{l, \tau'(s)}),$$

and an appeal to Lemma 15 yields that $Sz(K, \phi^\nu/2) > \omega^\xi$. This contradiction finishes the proof. To see that $h_{l, \tau'(s)} \in \cup_{i=1}^\nu N_{\theta^i, \phi^i, \Gamma_\xi}$, we note that if $h_{l, \tau'(s)} \in N_{\theta^i, \phi^i, \Gamma_\xi}$,

$$\varepsilon \leq h_{l, \tau'(t)}(e_{r(\sigma \circ \sigma'(s))}) \leq \|h_{l, \tau'(t)}\|_\infty \leq \theta^i.$$

This shows that $i \leq \nu$ by our choice of ν . □

Lemma 17. Fix $1 < \alpha, \beta < \infty$ and $0 < \phi < 2^{-1/\alpha}$ and assume that $1/\alpha + 1/\beta = 1$. Assume that for some $C \geq 1$ and all $i \in \mathbb{N}$, $Sz_\xi(K, \phi^i/2) \leq C2^i$. Let $\theta = 2^{-1/\alpha}$. Then for any $n \in \mathbb{N}$ and any $C_1 > C$, Player I has a winning strategy in the game with target set

$$\left\{ t \in \text{MAX}(\Gamma_{\xi,n}.X.C) : \left\| \sum_{s \preceq t} \mathbb{P}_{\xi,n}(s) e_s \right\|_{\theta, \phi, \alpha, \Gamma_{\xi,n}} > C_1 n^{1/\beta} \right\}.$$

Proof. Suppose not. Then for some $n \in \mathbb{N}$, there exist $(C_s)_{s \in \Gamma_{\xi,n}.X} \subset \mathcal{C}$ and

$$(f_t)_{t \in \text{MAX}(\Gamma_{\xi,n}.X)} \subset M_{\theta, \phi, \alpha, \Gamma_{\xi,n}}$$

such that

$$Cn^{1/\beta} < \inf_{t \in \text{MAX}(\Gamma_{\xi,n}.X)} f_t \left(\sum_{s \preceq t} \mathbb{P}_{\xi,n}(s) e_{r(s)} \right).$$

We may assume as in Lemma 16 that $\text{supp}(f_t) \subset [\preceq r(t)]$ for each $t \in \text{MAX}(\Gamma_{\xi,n}.X)$. Then by Theorem 4(i), there exist a level preserving extended pruning $(\sigma, \tau) : \Gamma_{\xi,n} \rightarrow \Gamma_{\xi,n}$ and numbers, b_1, \dots, b_n such that $Cn^{1/\beta} < \sum_{i=1}^n b_i$ and for all $1 \leq i \leq n$ and all $\Lambda_{\xi,n,i} \ni s \preceq t \in \text{MAX}(\Gamma_{\xi,n})$, $f_{\tau(t)}(e_{r(\sigma(s))}) \geq b_i$. Fix $\delta > 0$ such that $Cn^{1/\beta} + n\delta < \sum_{i=1}^n b_i$. Let $R = \{i \leq n : b_i \geq \delta\}$.

Sublemma 18. *There exist a level preserving extended pruning $(\sigma_0, \tau_0) : \Gamma_{\xi, n} \cdot X \rightarrow \Gamma_{\xi, n} \cdot X$, $l, w \in \mathbb{N}$, $(a_i)_{i=1}^l \in B_{\ell_\alpha}^l$, $(k_i)_{i \in R} \subset \{1, \dots, l\}$, $(w_i)_{i=1}^l \subset \{1, \dots, w\}$, and $(g_t)_{t \in \text{MAX}(\Gamma_{\xi, n} \cdot X)} \subset M_{\theta, \phi, \alpha, \Gamma_{\xi, n}}$ such that*

- (i) *for each $t \in \text{MAX}(\Gamma_{\xi, n} \cdot X)$, $\|g_t - f_{\tau \circ \tau_0(t)}\|_\infty < \delta$,*
- (ii) *for any $t \in \text{MAX}(\Gamma_{\xi, n} \cdot X)$, there exist disjointly supported functionals $h_{1,t}, \dots, h_{l,t}$ such that $h_{i,t} \in N_{\theta^{w_i}, \phi^{w_i}, \Gamma_{\xi, n}}$ and $g_t = \sum_{i=1}^l a_i h_{i,t}$,*
- (iii) *for $i \in R$ and $\Lambda_{\xi, n, i} \ni s \preceq t \in \text{MAX}(\Gamma_{\xi, n} \cdot X)$, $r(\sigma \circ \sigma_0(s)) \in \text{supp}(h_{k_i, t})$,*

We first finish the proof of the lemma and then return to the proof of the sublemma. Note that item (iii) of the sublemma implies that for $i \in R$ and $\Lambda_{\xi, n, i} \ni s \preceq t \in \text{MAX}(\Gamma_{\xi, n})$,

$$b_i \leq g_t(e_{r(\sigma \circ \sigma_0(s))}) + \delta = a_{k_i} \theta^{w_{k_i}} + \delta.$$

From this and our choice of δ we deduce that

$$Cn^{1/\beta} + \delta n < \sum_{i=1}^n b_i \leq \delta n + \sum_{i \in R} a_{k_i} \theta^{w_{k_i}}.$$

Partition R into sets R_1, \dots, R_l , where $R_j = \{i \in R : k_i = j\}$, so that

$$Cn^{1/\beta} < \sum_{i \in R} a_{k_i} \theta^{w_{k_i}} = \sum_{j=1}^l a_j \theta^{w_j} |R_j|.$$

We claim that for each j , $|R_j| \leq C2^{w_j}$. Indeed, suppose $|R_j| > C2^{w_j}$ for some j . By Theorem 4(iv), if $R_j = \{r_1, \dots, r_m\}$, with $r_1 < \dots < r_m$, there exists extended pruning $(\sigma', \tau') : \Gamma_{\xi, m} \cdot X \rightarrow \Gamma_{\xi, n} \cdot X$ such that $\sigma'(\Lambda_{\xi, m, i}) \subset \Lambda_{\xi, n, r_i}$. We now use Lemma 15 to deduce that $Sz_\xi(K, \phi^{w_j}/2) > C2^{w_j}$, which is a contradiction. Thus we deduce that $|R_j| \leq C2^{w_j}$ for each j . This means that for each $1 \leq j \leq l$,

$$\theta^{w_j} = (2^{-1/\alpha})^{w_j} = (2^{w_j})^{-1/\alpha} \leq C^{1/\alpha} |R_j|^{-1/\alpha} \leq C |R_j|^{-1/\alpha}.$$

Then

$$\begin{aligned} \sum_{j=1}^l a_j \theta^{w_j} |R_j| &\leq C \sum_{j=1}^l a_j |R_j|^{1-1/\alpha} = C \sum_{j=1}^l a_j |R_j|^{1/\beta} \leq C \left(\sum_{j=1}^l |a_j|^\alpha \right)^{1/\alpha} \left(\sum_{j=1}^l |R_j| \right)^{1/\beta} \\ &\leq C |R|^{1/\beta} \leq Cn^{1/\beta}. \end{aligned}$$

Thus we reach a contradiction.

We now return to the proof of the sublemma. First fix $w \in \mathbb{N}$ such that $\theta^w < \delta$. For each $t \in \text{MAX}(\Gamma_{\xi, n})$, write $f_{\tau(t)} = \sum_{i=1}^{k_t} a_{i,t} f_{i,t}$ for some disjointly supported $f_{i,t} \in \bigcup_{j=1}^\infty N_{\theta^j, \phi^j, \Gamma_{\xi, n}}$ and $a_{i,t} \geq 0$ such that $\sum_{i=1}^{k_t} a_{i,t}^\alpha \leq 1$. Let $S_t = \{i \leq k_t : \|a_{i,t} f_{i,t}\|_\infty \geq \delta\}$. Note that since $\sum_{i=1}^{k_t} a_{i,t}^\alpha \leq 1$, $|S_t| \leq \lfloor 1/\delta^\alpha \rfloor =: k_0$. As in the previous lemma, we write $\sum_{i \in S_t} a_{i,t} f_{i,t} = \sum_{i=1}^{l_t} a'_{i,t} f'_{i,t}$ for some $l_t \leq k_0$. Considering the function from $\text{MAX}(\Gamma_{\xi, n} \cdot X)$ given by $t \mapsto l_t \in \{1, \dots, k_0\}$, we use Theorem 4(iii) to obtain $l \in \mathbb{N}$ and a level preserving extended pruning $(\sigma', \tau') : \Gamma_{\xi, n} \cdot X \rightarrow \Gamma_{\xi, n} \cdot X$ such that for all $t \in \text{MAX}(\Gamma_{\xi, n} \cdot X)$, $l_{\tau'(t)} = l$. Note that since $\|a'_{i, \tau'(t)} f'_{i, \tau'(t)}\|_\infty \geq \delta$ for every $1 \leq i \leq l$ and $t \in \text{MAX}(\Gamma_{\xi, n})$, if $f'_{i, \tau'(t)} \in N_{\theta^j, \phi^j, \Gamma_{\xi, n}}$,

$j \leq w$. Let $w_{i,\tau'(t)}$ be the value $j \in \{1, \dots, w\}$ such that $f'_{i,\tau'(t)} \in N_{\theta^j, \phi^j, \Gamma_{\xi,n}}$. By considering the map from $MAX(\Gamma_{\xi,n}.X)$ into $B_{\ell_\alpha} \times \{1, \dots, w\}^l$ given by

$$t \mapsto ((a_{i,\tau'(t)})_{i=1}^l, (w_{i,\tau'(t)})_{i=1}^l),$$

we use Theorem 4(iii) again to find another level preserving extended pruning $(\sigma'', \tau'') : \Gamma_{\xi,n}.X \rightarrow \Gamma_{\xi,n}.X$, $(a_i)_{i=1}^l \in B_{\ell_\alpha}$ and $(w_i)_{i=1}^l \subset \{1, \dots, w\}$ such that for all $t \in MAX(\Gamma_{\xi,n})$, $\|(a_{i,\tau' \circ \tau''(t)})_{i=1}^l - (a_i)_{i=1}^l\|_{\ell_\alpha} < \delta$ and for all $1 \leq i \leq l$, $f'_{i,\tau' \circ \tau''(t)} \in N_{\theta^{w_i}, \phi^{w_i}, \Gamma_{\xi,n}}$. Note that for all $t \in MAX(\Gamma_{\xi,n}.X)$,

$$\|f_{\tau \circ \tau' \circ \tau''(t)} - \sum_{i=1}^l a_i f'_{i,\tau' \circ \tau''(t)}\|_\infty < \delta.$$

This implies that for any $i \in R$ and any $\Lambda_{\xi,n,i} \ni s \preceq t$, since

$$\delta \leq b_i \leq f_{\tau \circ \tau' \circ \tau''}(e_{r(\sigma \circ \sigma' \circ \sigma''(s))}),$$

$r(\sigma \circ \sigma' \circ \sigma''(s)) \in \bigcup_{j=1}^l \text{supp}(f'_{j,\tau' \circ \tau''(t)})$. Thus we may let $\kappa(s, t)$ be the unique $j \in \{1, \dots, l\}$ such that $r(\sigma \circ \sigma' \circ \sigma''(s)) \in \text{supp}(f'_{j,\tau' \circ \tau''(t)})$ if $s \in \bigcup_{i \in R} \Lambda_{\xi,n,i}$, and $\kappa(s, t) = 0$ otherwise. Applying Theorem 4(ii), we deduce the existence of $(k_i)_{i \in R} \subset \{1, \dots, l\}$ and a level preserving extended pruning $(\sigma''', \tau''') : \Gamma_{\xi,n}.X \rightarrow \Gamma_{\xi,n}.X$ such that setting $\sigma_0 = \sigma' \circ \sigma'' \circ \sigma'''$, $\tau_0 = \tau' \circ \tau'' \circ \tau'''$, $h_{i,t} = f'_{i,\tau_0(t)}$, and $g_t = \sum_{i=1}^l a_i h_{i,t}$, finishes the proof. \square

4. PROOF OF THE MAIN RESULTS

Proof of Theorem 1. Let $\phi = 1/3$ and $\theta = 2/3$, so that $\frac{1}{\theta - \phi} = 3$. Fix $1 < p < \infty$. Fix any $1 < \alpha, \beta < \infty$ such that $\beta < p$ and $1/\alpha + 1/\beta = 1$. Let $C' = C'(\beta, p)$ be the constant from Theorem 11. Fix $\varepsilon > 0$. By Lemma 16, Player I has a winning strategy in the game with target set

$$\left\{ t \in MAX(\Gamma_\xi.X.C) : \left\| \sum_{s \preceq t} \mathbb{P}_\xi(s) e_s \right\|_{\theta, \phi, \alpha, \Gamma_\xi} > \varepsilon \right\}.$$

By Corollary 14, for any B -tree T with $o(T) = \omega^{1+\varepsilon}$ and any normally weakly null collection $(f_t)_{t \in T.L_p(X)} \subset B_{L_p(X)}$,

$$\inf \left\{ \|\varrho(f)\|_{L_p} : t \in T.L_p(X), f \in \text{co}(f_s : \emptyset \prec s \preceq t) \right\} \leq 3C'\varepsilon.$$

We deduce $Sz(K_p) \leq \omega^{1+\varepsilon}$ by Theorem 5(i).

It is clear that $Sz(K) \leq Sz(K_p)$ for any $1 < p < \infty$. If K is convex, then either $Sz(K) = \infty$, in which case $Sz(K_p) = \infty = \omega^\infty = Sz(K)$, or there exists an ordinal ξ such that $Sz(K) = \omega^\xi$ [2, Proposition 4.2]. We deduce that $Sz(K_p) \leq \omega^{1+\varepsilon} = \omega Sz(K)$ by the previous paragraph. In the case that $\xi \geq \omega$, $1 + \xi = \xi$. \square

Proof of Theorem 3. If $\mathbf{p}_\xi(K) = \infty$, there is nothing to show, so assume $\mathbf{p}_\xi(K) < \infty$. Fix $1 < p, q < \infty$ with $1/p + 1/q = 1$. Fix $1 < \alpha, \beta, \gamma < \infty$ such that $\max\{\mathbf{p}_\xi(K), q\} < \gamma < \alpha$ and $1/\alpha + 1/\beta = 1$. Let $C' = C'(\beta, p)$ be the constant from Theorem 11. Let $\phi = 2^{-1/\gamma}$ and note that $\sup_{i \in \mathbb{N}} \varepsilon^\gamma S z_\xi(K, \phi^i/2)/2^i < \infty$. By Lemma 17, with $\theta = 2^{-1/\alpha}$, there exists a constant C_1 such that for every $n \in \mathbb{N}$, Player I has a winning strategy in the game with target set

$$\left\{ t \in \text{MAX}(\Gamma_{\xi,n}, X.C) : \left\| \sum_{s \preceq t} \mathbb{P}_{\xi,n}(s) e_s \right\|_{\theta, \phi, \alpha, \Gamma_{\xi,n}} > C_1/n^{1/\beta} \right\}.$$

By Corollary 14, for every $n \in \mathbb{N}$, every B -tree T with $o(T) = \omega^{1+\xi}n$, and every normally weakly null $(f_t)_{t \in T.L_p(X)} \subset B_{L_p(X)}$,

$$\inf \left\{ \| \varrho(f) \|_{L_p} : t \in T.L_p(X), f \in \text{co}(\emptyset \prec s \preceq t) \right\} \leq \frac{C_1 C'}{n(\theta - \phi)} n^{1/\beta} = \frac{C_1 C'}{n^{1/\alpha\theta - \phi}}.$$

By Theorem 5(ii), $\mathbf{p}_{1+\xi}(K_p) \leq \alpha$. Since $\alpha > \max\{\mathbf{p}_\xi(K), q\}$ was arbitrary, we deduce that $\mathbf{p}_{1+\xi}(K_p) \leq \max\{\mathbf{p}_\xi(K), q\}$. □

REFERENCES

- [1] P.A.H. Brooker, *Asplund operators and the Szlenk index*, Operator Theory 68 (2012), 405-442.
- [2] R.M. Causey, *An alternate description of the Szlenk index with applications*, Illinois J. Math. 59 (2) (2015), 359-390.
- [3] R.M. Causey, *The Szlenk index of injective tensor products and convex hulls*, to appear in Journal of Functional Analysis. DOI 10.1016/j.jfa.2016.12.017.
- [4] R.M. Causey, *Power type ξ -asymptotically uniformly smooth norms*, submitted.
- [5] R.M. Causey, *A note on the relationship between the Szlenk and w^* -dentability indices of arbitrary w^* -compact sets*, to appear in Positivity.
- [6] R.M. Causey, S.J. Dilworth, *ξ -asymptotically uniformly smooth, ξ -asymptotically uniformly convex, and (β) operators*, submitted.
- [7] 4] J. Lindenstrauss, L. Tzafriri, *Classical Banach spaces. II*, Ergebnisse der Mathematik und ihrer Grenzgebiete, Vol. 92. Springer-Verlag, Berlin, 1977.
- [8] P. Hájek, Th. Schlumprecht, *The Szlenk index of $L_p(X)$* , Bull. Lond. Math. Soc. 46 (2014), no. 2, 415-424.